Exact Differential Equations

The following type of first order differential equations that we'll be looking at is exact differential equations. What's exact differential equation?

Property: $M(x,y) + N(x,y)y' = 0$ or $M(x,y)dx + N(x,y)dy = 0$
(1)

is an **exact** differential equation if and only if

$$M_y(x,y) = N_x(x,y)$$

For exact differential equation (1), there exists a function $\varphi(x,y)$such that $\varphi_x(x,y) = M(x,y)$ and $\varphi_y(x,y) = N(x,y)$. Thus, $M(x,y) + N(x,y)y' = \varphi_x(x,y) + \varphi_y(x,y)y' = \frac{d}{dx}\varphi(x,y)$

Thus, differential equation (1) becomes

$$\frac{d}{dx}\varphi(x,y) = 0$$

So, $\varphi(x,y) = c$ is an implicit solution of differential equation (1).

Let's look at the detail procedure to find the solution for exact differential equation from the following examples.

Example 1: Solve the differential equation:

$$(ycosx + 2xe^y) + (sinx + x^2e^y - 1)y' = 0$$

Let $M = y\cos x + 2xe^y, N = \sin x + x^2e^y - 1$

$M_y = \cos x + 2xe^y, N_x = \cos x + 2xe^y$

$M_y = N_x$

$\therefore$ There exists a function φ, such that $\varphi_x = M, \varphi_y = N$.

$$\begin{cases} \varphi_x = y\cos x + 2xe^y & (a) \\ \varphi_y = \sin x + x^2e^y - 1 & (b) \end{cases}$$

Integrating (a) with respect to x, we have

$\varphi = \int y\cos x + 2xe^y dx = y\sin x + x^2e^y + h(y)$

Take dirivative with respect to y, we have

$\varphi_y = \sin x + x^2e^y + h'(y)$

From (b), we have $\varphi_y = \sin x + x^2 e^y - 1,$ so

$\sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1,$

$h'(y) = -1$

$h(y) = -y + c$

Thus, $\varphi = y \sin x + x^2 e^y - y + c$

$y \sin x + x^2 e^y - y + c = 0$ is the solution to

$y \cos x + 2xe^y + (\sin x + x^2 e^y - 1)y' = 0$

Example 2: Solve the differential equation:

$$xy^2 + 2 = (3 - x^2 y)y'$$

Transfer the original equation to

$(xy^2 + 2)dx + (x^2 y - 3)dy = 0$

Let $M = xy^2 + 2, N = x^2 y - 3$

$M_y = 2xy, N_x = 2xy$

$M_y = N_x$

$\therefore$ There exists a function φ, such that $\varphi_x = M, \varphi_y = N.$

$$\begin{cases} \varphi_x = xy^2 + 2 & (a) \\ \varphi_y = x^2 y - 3 & (b) \end{cases}$$

Integrating (a) with respect to $x,$ we have

$\varphi = \int (xy^2 + 2)dx = \frac{1}{2}x^2 y^2 + 2x + h(y)$

Take dirivative with respect to $y,$ we have

$\varphi_y = x^2 y + h'(y)$

From (b), we have $\varphi_y = x^2 y - 3,$ so

$x^2 y + h'(y) = x^2 y - 3,$

$h'(y) = -3$

$h(y) = -3y + c$

Thus, $\varphi = \frac{1}{2}x^2 y^2 + 2x - 3y + c$

$\frac{1}{2}x^2 y^2 + 2x - 3y + c = 0$ is the solution to

$xy^2 + 2 = (3 - x^2 y)y'$

Example 3: Solve the differential equation:

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2 + 1))y' = 0$$

with initial value: $y(1) = 0.$

Transfer the original equation to

$$(\frac{2ty}{t^2+1}-2t)dt+(\ln(t^2+1)-2)dy=0$$

Let $M=\frac{2ty}{t^2+1}-2t, N=\ln(t^2+1)-2$

$$M_y=\frac{2t}{t^2+1}, N_t=\frac{2t}{t^2+1}$$

$$M_y=N_x$$

$\therefore$ There exists a function φ, such that $\varphi_x=M, \varphi_y=N$.

$$\begin{cases} \varphi_x=\frac{2ty}{t^2+1}-2t & (a) \\ \varphi_y=\ln(t^2+1)-2 & (b) \end{cases}$$

Integrating (a) with respect to t, we have

$$\varphi=\int(\frac{2ty}{t^2+1}-2t)dt=y\ln(t^2+1)-t^2+h(y)$$

Take dirivative with respect to y, we have

$$\varphi_y=\ln(t^2+1)+h'(y)$$

From (b), we have $\varphi_y=\ln(t^2+1)-2$, so

$$\ln(t^2+1)+h'(y)=\ln(t^2+1)-2,$$

$$h'(y)=-2$$

$$h(y)=-2y+c$$

Thus, $\varphi=y\ln(t^2+1)-t^2-2y+c$

$y\ln(t^2+1)-t^2-2y+c=0$ is the solution to

$$(\frac{2ty}{t^2+1}-2t)dt+(\ln(t^2+1)-2)dy=0$$

$\because y(1)=0$

$\therefore 0\bullet\ln(1^2+1)-1^2-2\bullet 0+c=0 \Rightarrow c=1$

$\therefore y\ln(t^2+1)-t^2-2y-1=0$ is the solution to

$y\ln(t^2+1)-t^2-2y+c=0$ with initial value $y(1)=0$.

Remark: This equation is first order linear, you can use first order linear equation method to solve it too.

Integrating Factor: Sometimes it is possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor.

Multiply $\mu(x)$ both sides to

$$M(x,y)dx+N(x,y)dy=0,$$

so that the resulting equation

$$\mu(x)M(x,y)dx+\mu(x)N(x,y)dy=0$$

is exact, then we call $\mu(x)$ is an integrating factor.

Example 4: Verify that $\mu(x) = x$ is an integrating factor of the equation

(2) $$(3xy + y^2) + (x^2 + xy)y' = 0$$

Multiplying $\mu(x) = x$ to both sides of (2), we have

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0$$

Let $M = 3x^2y + xy^2, N = x^3 + x^2y,$ we have

$$M_y = 3x^2 + 2xy, N_x = 3x^2 + 2xy$$
$$\therefore M_y = N_x$$

Differential Equation (2) is exact.

Example 5: Given $\mu(x) = x$ is an integrating factor of the equation

(2) $$(3xy + y^2) + (x^2 + xy)y' = 0$$

Find the solution to (2).

Multiplying $\mu(x) = x$ to both sides of (2), we have

(3) $$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0$$

(3) is an exact equation, there exists a function φ, such that

$$\varphi_x = M = 3x^2y + xy^2, \varphi_y = N = x^3 + x^2y$$

$$\varphi = \int 3x^2y + xy^2 dx = x^3y + \frac{1}{2}x^2y^2 + h(y)$$
$$\varphi_y = x^3 + x^2y + h'(y)$$
$$\because \varphi_y = x^3 + x^2y$$
$$\therefore x^3 + x^2y + h'(y) = x^3 + x^2y$$
$$h'(y) = c$$
$$\therefore \varphi = x^3y + \frac{1}{2}x^2y^2 + c$$

$\therefore x^3y + \frac{1}{2}x^2y^2 = 0$ is the solution to $(3xy + y^2) + (x^2 + xy)y' = 0.$